7.2 Partial Derivatives

How do we define the derivative of a function in more than one variable?

One way is to look just at individual variables

$f(x,y)$ is a function of two varibles

$\frac{\partial f}{\partial x}$ is the partial derivative of f with respect to x

"how f changes as x changes"

$\frac{\partial f}{\partial y}$ is the partial derivative of f with respect to y

"how f changes as y changes"

To compute $\frac{\partial f}{\partial x}$ think of x as a variable and y as a constant

ex/ $f(x,y) = 5x^3y^2$ to compute $\frac{\partial f}{\partial x}$

$f(x,y) = [5y^2]x^3$

treat as constant

$$\frac{\partial f}{\partial x} = 3[5y^2]x^2 = 15x^2y^2$$

$$\frac{\partial f}{\partial y} = 10x^3y$$

Other notation: $\frac{\partial f}{\partial x} = f_x$

$\frac{\partial f}{\partial y} = f_y$

$\frac{\partial f}{\partial z} = f_z$ . . .

ex/

$$f(x,y) = 3x^2 + 2xy + 5y$$

$\frac{\partial f}{\partial x}$ ~~treat~~ $f(x,y) = 3x^2 + [2y]x + [5y]$

treat as constants

$$\frac{\partial f}{\partial x} = 6x + 2y$$

$\frac{\partial f}{\partial y}$ $f(x,y) = [3x^2] + [2x]y + 5y$

treat as constants

$$\frac{\partial f}{\partial y} = 2x + 5$$

ex/

$$f(x,y) = 6x^3 + 5xy^2 + 4y^5 + 2x^2y + 8$$

$$\frac{\partial f}{\partial x} = 18x^2 + 5y^2 + 4xy$$

$$\frac{\partial f}{\partial y} = 10xy + 20y^4 + 2x^2$$

ex/

$$f(x,y) = 3(2x^2 + 3y)^3$$

w.r.t. x: $f(x,y) = 3(2x^2 + [3y])^3$

treat as constant

$$\frac{\partial f}{\partial x} = 9(2x^2+3y)^2(4x)$$

$$\frac{\partial f}{\partial y} = 9(2x^2+3y)^2(3)$$

ex/

$$f(x,y) = \frac{x}{2x-3y}$$

quotient rule

$$f_x = \frac{(2x-3y)(1) - (x)(2)}{(2x-3y)^2}$$

$$f_y = \frac{(2x-3y)(0) - x(-3)}{(2x-3y)^2}$$

same

using product rule

$$f(x,y) = x(2x-3y)^{-1}$$

$$f_y = -x(2x-3y)^{-2}(-3)$$

$$= \frac{3x}{(2x-3y)^2}$$

ex/

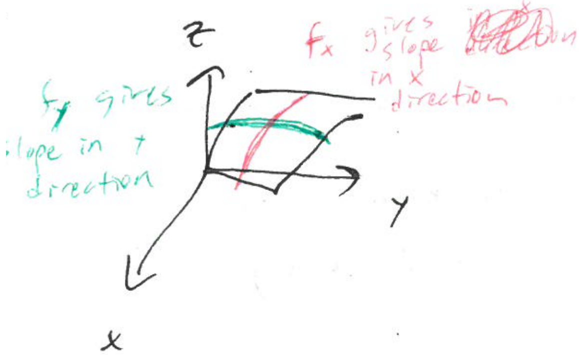
$$f(x,y) = ye^{xy}$$

$$f_x = ye^{xy}(y) = y^2e^{xy}$$

$$f_y = e^{xy} + ye^{xy}(x)$$

Geometric Interpretation:

Recall $z = f(x,y)$ is a surface

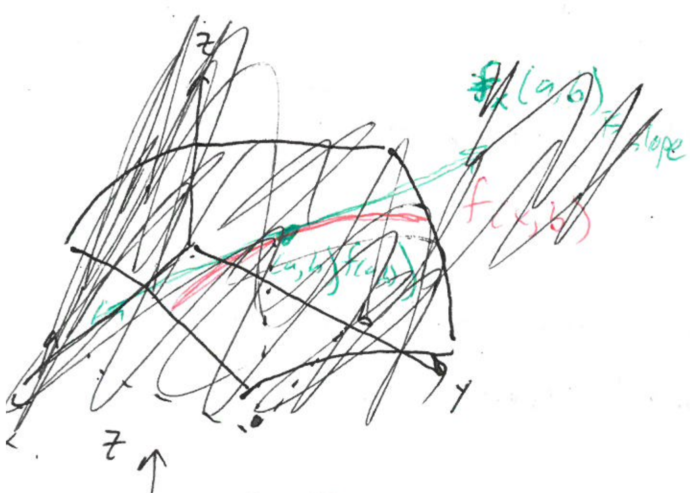


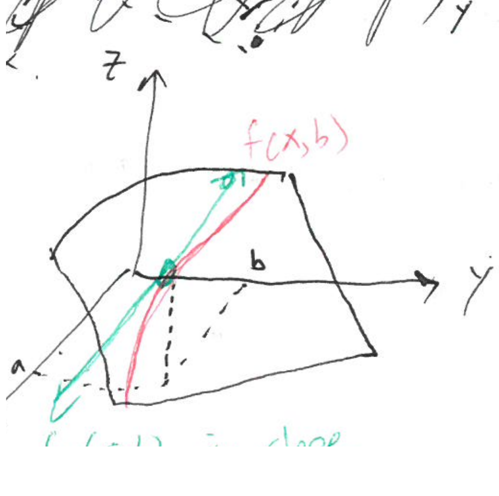
If we hold y constant and allow x to vary

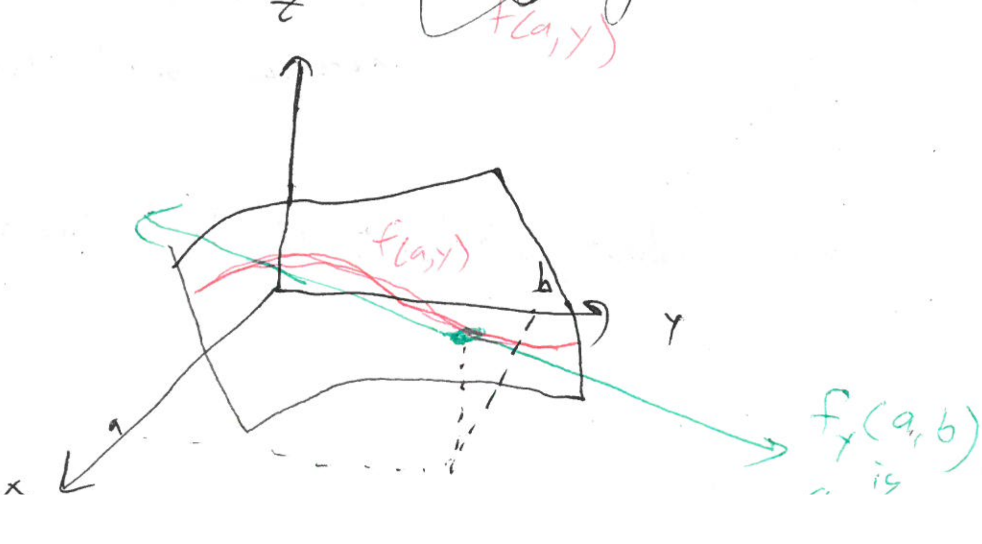
$z = f(x,b)$ describes a curve on the surface

and $\frac{\partial f}{\partial x}(a,b)$ is the slope of this curve when $x = a$

Similarily, if we hold x constant at a and let y vary then $z = f(a,y)$ gives a curve on the surface







If at a point $\frac{\partial f}{\partial x} > 0$ then f is increasing as we move in the positive x direction (or decreasing as x decreases)

If $\frac{\partial f}{\partial x} < 0$ then f is decreasing as we move in the positive x-direction. (or increasing as x decreases)

x

$$f(x,y) = 10\, x^{3/4}\, y^{1/4}$$

x is labor input
y is capital input
f is production output

Find $f_x(1,16)$
and $f_y(1,16)$
and interpret

$$f_x = \frac{30}{4} x^{-1/4} y^{1/4} \qquad f_x(1,16) = \frac{30}{4} \cdot 2 = 15$$

$$f_y = \frac{5}{2} x^{3/4} y^{-3/4} \qquad f_y(1,16) = \frac{5}{2} \cdot 1 \cdot \frac{1}{8} = \frac{5}{16}$$

$f_x(1,16) > 0$
$f_y(1,16) > 0$

so increasing labor and capital increases production

but ~~where~~ what gives best increase of production?

$f_x(1,16) = 15$ so production $\uparrow$ by 15 for increase by 1 of labor

$f_y(1,16) = \frac{5}{16}$ so production $\uparrow$ by $\frac{5}{16}$ for increase by 1 of capital

$\Rightarrow$ better off spending more in labor costs

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ex/ The Volume of a certain gas is determined by the temperature (T) and Pessure (P) by

$$V = .08\,(T/P)$$

Calculate and interpret $\frac{\partial V}{\partial P}$ and $\frac{\partial V}{\partial T}$ when $P = 20$, $T = 300$

$$\frac{\partial V}{\partial P} = -.08\,(T/P^2), \quad \frac{\partial V}{\partial P}(20,300) = -.08\left(\frac{300}{20^2}\right) = -0.06$$

$$\frac{\partial V}{\partial T} = .08\,(1/P), \quad \frac{\partial V}{\partial T}(20,300) = 0.08\left(\frac{1}{20}\right) = 0.004$$

$\frac{\partial V}{\partial P} < 0$ so as pressure increases, Volume decreases.

$\frac{\partial V}{\partial T} > 0$ so as temperture increases, volume increases.

Higher order Partial Derivatives

1st derivatives — 2nd derivatives

$$f(x,y) \to f_x(x,y) = \frac{\partial f}{\partial x} \to \begin{cases} f_{xx}(x,y) = \dfrac{\partial^2 f}{\partial x^2} \\ f_{xy}(x,y) = \dfrac{\partial^2 f}{\partial y\, \partial x} \end{cases}$$

$$f(x,y) \to f_y(x,y) = \frac{\partial f}{\partial y} \to \begin{cases} f_{yx}(x,y) = \dfrac{\partial^2 f}{\partial x\, \partial y} \\ f_{yy}(x,y) = \dfrac{\partial^2 f}{\partial y^2} \end{cases}$$

4 2nd or. derivati

ex

$$f(x,y) = e^{x^2y} + 2x^2 + 3y^3 + 2$$

$$f_x(x,y) = 2x\, e^{x^2y} + 4x \to f_{xx} = 2e^{x^2y} + 2x(2x)e^{x^2y} + 4$$

$$\to f_{xy} = 2x\, e^{x^2y}$$

same!

$$f_y(x,y) = e^{x^2y} + 9y^2 \to f_{yx} = 2x\, e^{x^2y}$$

$$\to f_{yy} = e^{x^2y} + 18y$$

ex

$$f(x,y) = 2x^3y + 7x^2 - 4y + 1$$

$$f_x = 6x^2y + 14x \to f_{xx} = 12xy + 14$$

$$\to f_{xy} = 6x^2$$

same!

$$f_y = 2x^3 - 4 \to f_{yx} = 6x^2$$

$$\to f_{yy} = 0$$

for "nice" functions

$f_{xy} = f_{yx}$